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Remarks on topological SUSY in six-dimensional TQFTs

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Abstract: We establish the existence of the topological vector supersymmetry in the six dimensional topological field theory for two-form fields introduced by Baulieu and West. We investigate the relation of these symmetries to the twist operation for the $(2,0)$ supersymmetry and comment on their resemblance to the analogous symmetries in topological Yang-Mills theory.

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1. Introduction

There has been recently some interest in higher dimensional topological field theories. In particular Baulieu and West studied a six dimensional topological model of two-form fields in [1]. They also discussed the BRST quantization of this model and a new kind of topological twist operation specific to $(2, 0)$ supersymmetry in six dimensions. The aim of this paper is to elaborate on their work and investigate closer the BRST-structure of the model and its relation to a twist of the $(2, 0)$ supersymmetry. More specifically we are looking for a completion of the topological supersymmetry. In [1] the authors showed that the part of the BRST-operator that encodes the topological shift symmetry can be thought of being obtained by twisting the $(2, 0)$ Poincaré supersymmetry algebra. The complete BRST operator is built up by promoting also the usual two-form field gauge symmetries to a BRST-symmetry. All this structure is completely parallel to what happens in four dimensional topological Yang-Mills theory [2, 3].

In the latter theory it has been shown that there also exists a fermionic vectorial supersymmetry whose anti-commutator with the BRST-operator closes onto translation [4]. Of course, the existence of this additional symmetry can to some extent be motivated from the fact that the twist of the $N = 2$ supersymmetry in four dimensions also gives rise to a vectorial supersymmetric partner of the BRST-charge [5]. It seems therefore natural to ask also for such a vector supersymmetry in the topological model of [1]. It is this question which we will investigate in this paper.

The paper is organized as follows. In section two we will present the model and discuss the BRST action on the gauge field sector. In section three we present the gauge fixing of the shift symmetry Q and the symmetry s. Additionally, we rewrite the Q-fixed action in a new basis for later use. In the next section we establish the existence of the topological supersymmetry and discuss the twist operation of the $(2,0)$ supersymmetry. We obtain the Q-operator related to the topological shift symmetry and also a vector operator. In this context we point out that the form of the vectorial transformation laws are in close analogy to the vectorial transformation laws obtained by twisting the four dimensional $N=2$ SYM theory. Concluding, we state why the transformations derived from the twisting procedure do not close onto translations modulo gauge-transformations and equations of motion.

2. Gauge field and ghosts

For the construction of a TQFT in six dimensions one needs two-form fields $B_2 = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu$ and ${}^cB_2 = \frac{1}{2}{}^cB_{\mu\nu}dx^\mu \wedge dx^\nu$ constrained by the self-duality conditions between these two independent real two-forms

$$\partial_{[\mu}B_{\nu\rho]} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}\partial^{[\alpha\,c}B^{\beta\gamma]} = 0\,, \tag{2.1}$$

in order to get a topological invariant action Σ_{cl} which does not vanish trivially

$$\Sigma_{cl} = \int F_3 \wedge {}^cF_3\,. \tag{2.2}$$

The corresponding field strength tensors are defined by $F_3 = dB_2$ and ${}^cF_3 = d{}^cB_2$, where $d = dx^\mu\partial_\mu$ denotes the exterior derivative. The action (2.2) possesses two gauge symmetries

$$\delta_{\lambda_2}B_2 = \lambda_2 \text{ and } \delta_{\lambda_1}B_2 = d\lambda_1\,, \tag{2.3}$$

and similar equations for ${}^cB_2$. λ_2 and λ_1 are a two-form and a one-form characterizing the infinitesimal gauge symmetry.

The usual BRST-quantization demands that these infinitesimal gauge parameters are transformed into ghost fields ψ_2^1 and V_1^1.

Corresponding to eq. (2.3) the total BRST operator is therefore defined by

$$\underline{s} = Q + s\,, \tag{2.4}$$

where Q represents the topological shift symmetry [2] and s is the usual BRST-symmetry induced by the gauge freedom of the model. Generalizing the concepts of [6] the total BRST-symmetry is constructed by the so called horizontality conditions

$$(\underline{s} + d)\left(B_2 + V_1^1 + m^2\right) = F_3 + \psi_2^1 + \phi_1^2 + \phi^3\,, \tag{2.5}$$

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which implies the horizontality condition

$$(\underline{s}+d)\left(F_3+\psi_2^1+\phi_1^2+\phi^3\right)=0\,, \tag{2.6}$$

by virtue of the nilpotency of $(\underline{s}+d)$. Eq. (2.5) yields in terms of forms

$$\begin{aligned} F_3 &= dB_2\,, \\ \underline{s}B_2 &= \psi_2^1 - dV_1^1\,, \\ \underline{s}V_1^1 &= \phi_1^2 - dm^2\,, \\ \underline{s}m^2 &= \phi^3\,, \end{aligned} \tag{2.7}$$

whereas (2.6) leads to

$$\begin{aligned} dF_3 &= 0\,, \\ \underline{s}F_3 &= -d\psi_2^1\,, \\ \underline{s}\psi_2^1 &= -d\phi_1^2\,, \\ \underline{s}\phi_1^2 &= -d\phi^3\,, \\ \underline{s}\phi^3 &= 0\,. \end{aligned} \tag{2.8}$$

The transformations in component notation for $B_{\mu\nu}, \psi^1_{\mu\nu}, V^1_\mu, \phi^2_\mu, m^2, \phi^3$ are deduced from (2.7) and (2.8):

$$\begin{aligned} \underline{s}B_{\mu\nu} &= \psi^1_{\mu\nu} + 2\partial_{[\mu}V^1_{\nu]}\,, \\ \underline{s}\psi^1_{\mu\nu} &= -2\partial_{[\mu}\phi^2_{\nu]}\,, \\ \underline{s}V^1_\mu &= \phi^2_\mu - \partial_\mu m^2\,, \\ \underline{s}\phi^2_\mu &= \partial_\mu\phi^3\,, \\ \underline{s}m^2 &= \phi^3\,, \\ \underline{s}\phi^3 &= 0. \end{aligned} \tag{2.9}$$

One identifies the non-vanishing Q-symmetry by

$$\begin{aligned} QB_{\mu\nu} &= \psi^1_{\mu\nu}\,, \\ Q\psi^1_{\mu\nu} &= -2\partial_{[\mu}\phi^2_{\nu]}\,, \\ Q\phi^2_\mu &= \partial_\mu\phi^3\,, \end{aligned} \tag{2.10}$$

and the non-vanishing BRST-symmetry stemming from the gauge freedom is given through

$$\begin{aligned} sB_{\mu\nu} &= 2\partial_{[\mu}V^1_{\nu]}\,, \\ sV^1_\mu &= -\partial_\mu m^2\,. \end{aligned} \tag{2.11}$$

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The total BRST operator $\underline{s}$ and the exterior derivative d acting on the above fields are both nilpotent and fulfill therefore

$$\{\underline{s}, d\} = d^2 = \underline{s}^2 = 0. \tag{2.12}$$

The dimensions and the corresponding $\Phi\Pi$-charges of the gauge fields and the ghosts are given in table 1.

	$B_{\mu\nu}$	V^1_μ	$\psi^1_{\mu\nu}$	ϕ^2_μ	m^2	ϕ^3
dim	2	1	2	1	0	0
$\Phi\Pi$	0	1	1	2	2	3

Table 1: Dimensions and Faddeev-Popov charges.

3. Gauge fixing procedure

3.1 The shift symmetry

In order to implement the self-duality condition (2.1) one introduces a general three-form $H_3 = \frac{1}{3!} H_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$ with its corresponding anti-ghost $\chi_3^{-1} = \frac{1}{3!}\chi^{-1}_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$. These fields are members of a BRST-doublet

$$Q\chi^{-1}_{\mu\nu\rho} = H_{\mu\nu\rho}\,, \qquad QH_{\mu\nu\rho} = 0\,. \tag{3.1}$$

For the gauge-fixing of the Q-symmetry one proceeds similarly by an introduction of the anti-ghost fields $\phi_\mu^{-2}, \phi^{-3}, X^1$ with their BRST-doublet partners (see also table 2)

$$\begin{aligned} Q\phi_\mu^{-2} &= \eta_\mu^{-1}\,, & Q\eta_\mu^{-1} &= 0\,, \\ Q\phi^{-3} &= \eta^{-2}\,, & Q\eta^{-2} &= 0\,, \\ QX^1 &= \eta^2\,, & Q\eta^2 &= 0\,. \end{aligned} \tag{3.2}$$

		$\psi^1_{\mu\nu}$		
	ϕ_μ^{-2}		ϕ^2_μ	
ϕ^{-3}		X^1		ϕ^3

Table 2: Topological ghosts and anti-ghosts.

In the spirit of [1, 7] we are now able to fix the topological shift symmetry with its reducible symmetries by the following gauge conditions à la BRST[1]

$$\begin{aligned} \Sigma_Q = \int d^6x\, \frac{1}{3!} Q\left\{\chi^{-1\mu\nu\rho}\left(3\partial_\mu B_{\nu\rho} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma} 3\partial^{\alpha c} B^{\beta\gamma} - \frac{\alpha_1}{2} H_{\mu\nu\rho}\right)\right\} + \\ + \int d^6x\, Q\left\{\phi_\mu^{-2}\partial_\nu\psi^{1\mu\nu} + \phi^{-3}\left(\alpha_2\eta^2 + \partial^\mu\phi^2_\mu\right) + X^1\left(\alpha_3\eta^{-2} + \partial^\mu\phi_\mu^{-2}\right) - \text{c.c.}\right\}\,. \end{aligned} \tag{3.3}$$

We also included additional couplings with parameters α_i which are consistent with the symmetries, dimensions and $\Phi\Pi$-charges. Later on, we will see that the topological vector SUSY places some restrictions on these parameters.

Using the definition of Q one can eliminate the field $H_{\mu\nu\rho}$ by using its algebraic equation of motion

$$H_{\mu\nu\rho} = \frac{1}{\alpha_1}\left(3\partial_{[\mu}B_{\nu\rho]} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma} 3\partial^{\alpha c} B^{\beta\gamma}\right)\,. \tag{3.4}$$

[1] In eq. (3.4) the notation cc means terms identical to the explicit ones, which are obtained by replacing all fields with their counterparts with index ^{c}.

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In this way one gets

$$
\begin{aligned}
\Sigma_Q = \int d^6x \, \Bigg\{ & \frac{1}{2\cdot 3!\cdot\alpha_1} \left| 3\partial_{[\mu}B_{\nu\rho]} + \frac{1}{6}3\varepsilon_{\mu\nu\rho\alpha\beta\gamma}\partial^{\alpha\,c}B^{\beta\gamma}\right|^2 - \\
& -\frac{1}{3!}\chi^{-1\mu\nu\rho}\left(3\partial_\mu\psi^1_{\nu\rho} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^{\alpha c}\psi^{1\beta\gamma}\right)\Bigg\} + \\
& + \int d^6x \, Q\left\{\phi_\mu^{-2}\partial_\nu\psi^{1\mu\nu} + \phi^{-3}\left(\alpha_2\eta^2 + \partial^\mu\phi^2_\mu\right) + X^1\left(\alpha_3\eta^{-2} + \partial^\mu\phi_\mu^{-2}\right) - \text{c.c.}\right\} .
\end{aligned}
\tag{3.5}
$$

In defining

$$
\Sigma^1_Q = \int d^6x \, \frac{1}{3!}\left| 3\partial_{[\mu}B_{\nu\rho]} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^{\alpha\,c}B^{\beta\gamma}\right|^2 , \tag{3.6}
$$

one gets for Σ^1_Q the following alternative expression

$$
\Sigma^1_Q = \int d^6x \, \frac{1}{3!}\left\{ |3\partial_\mu B_{\nu\rho}|^2 - |3\partial_\mu{}^c B_{\nu\rho}|^2 + \frac{2}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^\mu B^{\nu\rho}3\partial^{\alpha\,c}B^{\beta\gamma}\right\} . \tag{3.7}
$$

Eq. (3.7) allows to rewrite eq. (3.6) to

$$
\begin{aligned}
\Sigma_Q = \frac{1}{2\alpha_1}\Sigma^1_Q - & \int d^6x \, \frac{1}{3!}\left\{\chi^{-1\mu\nu\rho}\left(3\partial_\mu\psi^1_{\nu\rho} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^{\alpha c}\psi^{1\beta\gamma}\right)\right\} + \\
& + \int d^6x \, Q\left\{\phi_\mu^{-2}\partial_\nu\psi^{1\mu\nu} + \phi^{-3}\left(\alpha_2\eta^2 + \partial^\mu\phi^2_\mu\right) + X^1\left(\alpha_3\eta^{-2} + \partial^\mu\phi_\mu^{-2}\right) - \text{c.c.}\right\} .
\end{aligned}
\tag{3.8}
$$

Using Σ_{cl} of eq. (2.2) one can calculate the complete shift symmetry fixed action

$$
\begin{aligned}
\Sigma_{topo} &= -\frac{1}{\alpha_1}\Sigma_{cl} + \Sigma_Q \\
&= \int d^6x \, \Bigg\{ \frac{1}{2\cdot 3!\cdot\alpha_1}\left(|3\partial_\mu B_{\nu\rho}|^2 - |3\partial_\mu{}^c B_{\nu\rho}|^2\right) - \\
&\qquad -\frac{1}{3!}\chi^{-1\mu\nu\rho}\left(3\partial_\mu\psi^1_{\nu\rho} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^{\alpha c}\psi^{1\beta\gamma}\right)\Bigg\} + \\
&\quad + \int d^6x \, \Big\{\eta^{-1\mu}\partial^\nu\psi^1_{\mu\nu} + \eta^2\partial^\mu\phi_\mu^{-2} + \eta^{-2}\partial^\mu\phi^2_\mu - X^1\partial^\mu\eta_\mu^{-1} - 2\phi_\mu^{-2}\partial_\nu\partial^{[\mu}\phi^{2\nu]} - \\
&\qquad -\phi^{-3}\partial^\mu\partial_\mu\phi^3 + (\alpha_2+\alpha_3)\eta^{-2}\eta^2 - \text{c.c.}\Big\} ,
\end{aligned}
\tag{3.9}
$$

where we also evaluated the Q-variations with the help of (2.10) and (3.2). Using the algebraic equations of motion for the fields $\eta^{\pm 2}$

$$
\eta^{\pm 2} = -\frac{1}{\alpha_2+\alpha_3}\partial^\mu\phi^{\pm 2}_\mu , \tag{3.10}
$$

and with the choice of $\alpha_1 = 1$ and $\alpha_2 + \alpha_3 = 1$ one gets finally

$$
\begin{aligned}
\Sigma_{topo} = \int d^6x \, \Bigg\{ & \frac{1}{2\cdot 3!}\left(|3\partial_\mu B_{\nu\rho}|^2 - |3\partial_\mu{}^c B_{\nu\rho}|^2\right) - \\
& -\frac{1}{3!}\chi^{-1\mu\nu\rho}\left(3\partial_\mu\psi^1_{\nu\rho} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^{\alpha c}\psi^{1\beta\gamma}\right)\Bigg\} + \\
& + \int d^6x \, \left\{\eta^{-1\mu}\partial^\nu\psi^1_{\mu\nu} - X^1\partial^\mu\eta_\mu^{-1} - \phi^2_\mu\partial^2\phi^{-2\mu} + \phi^3\partial^2\phi^{-3} - \text{c.c.}\right\} .
\end{aligned}
\tag{3.11}
$$

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We would like to stress that our Q-fixed action differs from [1, eq. (2.23)]. It seems that this can be traced back to the following. In [1] the authors assumed the fields $\chi^{-1}_{\mu\nu\rho}$, $H_{\mu\nu\rho}$ to be self-dual (with ${}^c\chi_{\mu\nu\rho}$ and ${}^cH_{\mu\nu\rho}$ antiself-dual). Since in six dimensions the variation with respect to a self-dual field is antiself-dual one would need to introduce cross terms like $\chi_{\mu\nu\rho}{}^cH^{\mu\nu\rho} - c.c.$ in order to generate an equation of motion like (3.4). The introduction of such cross terms is however equivalent to our method of assuming $\chi^{-1}_{\mu\nu\rho}$ and $H_{\mu\nu\rho}$ not to fulfill (anti)self-duality conditions.

These mixing terms ($\chi^{\mu\nu\rho}\partial_\mu\psi_{\nu\rho}$ and $\chi^{\mu\nu\rho}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}\partial^{\alpha c}\psi^{1\beta\gamma}$) make it impossible to understand the action as the sum of two free six-form multiplets plus one decoupled part of purely topological packages.

The Q-transformations after the insertion of the algebraic equations of motion, equal those found in [1].

$$
\begin{aligned}
QB_{\mu\nu} &= \psi^1_{\mu\nu}\,, \\
Q\psi^1_{\mu\nu} &= -2\partial_{[\mu}\phi^2_{\nu]}\,, \\
Q\phi^2_\mu &= \partial_\mu\phi^3\,, \\
Q\phi^{-2}_\mu &= \eta^{-1}_\mu\,, \\
Q\eta^{-1}_\mu &= 0\,, \\
Q\phi^{-3} &= -\partial^\mu\phi^{-2}_\mu\,, \\
QX^1 &= -\partial^\mu\phi^2_\mu\,, \\
Q\chi^{-1}_{\mu\nu\rho} &= \left(3\partial_{[\mu}B_{\nu\rho]} + \frac{1}{6}\varepsilon_{\mu\nu\rho\alpha\beta\gamma}3\partial^{\alpha c}B^{\beta\gamma}\right).
\end{aligned}
\tag{3.12}
$$

Except for the equation involving $\chi^{-1}_{\mu\nu\rho}$ similar transformations are valid for the *c.c.*-sector.

For later use, we write the action and the Q-transformations in a different basis. In order to write the redefinitions as compact as possible we use the shorthand $\Phi = (F, B, \psi^1, \phi^{\pm2}, \phi^{\pm3})$:

$$
\begin{aligned}
\Phi_\pm &= \Phi \pm {}^c\Phi\,, \\
(\chi^{-1})^\pm &= \frac{1}{2}(1 \pm *)\chi^{-1}\,,
\end{aligned}
\tag{3.13}
$$

We denote the (anti-)selfdual parts of a three form by the superscripts $+(-)$, whereas the subscript $+(-)$, is related to the plus (minus) sign in the field redefinitions in (3.13). Rewriting the action (3.11) yields:

$$
\begin{aligned}
\Sigma_{\text{topo}} = \int d^6x \Big\{ &-\frac{1}{2\cdot 3!}3\partial_{[\mu}(B_+)_{\nu\rho]}3\partial^\mu(B_-)^{\nu\rho} - \\
&- \frac{1}{3!\cdot 3!}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}(\chi^-)^{-1}_{\mu\nu\rho}3\partial_\sigma(\psi^1_+)_{\tau\lambda} + \frac{1}{3!\cdot 3!}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}(\chi^+)^{-1}_{\mu\nu\rho}3\partial_\sigma(\psi^1_-)_{\tau\lambda}\Big\} + \\
&+ \int d^6x \frac{1}{2}\left\{(\eta_+)^{-1\mu}\partial^\nu(\psi^1_-)_{\mu\nu} - (X^1_+)\partial^\mu(\eta^{-1}_-)_\mu - (\phi^2_+)_\mu\partial^2(\phi^{-2}_-)^\mu + \phi^3_+\partial^2\phi^{-3}_-\right\} + \\
&+ \int d^6x \frac{1}{2}\left\{(\eta_-)^{-1\mu}\partial^\nu(\psi^1_+)_{\mu\nu} - (X^1_-)\partial^\mu(\eta^{-1}_+)_\mu - (\phi^2_-)_\mu\partial^2(\phi^{-2}_+)^\mu + \phi^3_-\partial^2\phi^{-3}_+\right\}.
\end{aligned}
\tag{3.14}
$$

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If one collects the fields into two the $+(-)$ sectors as $\{\Phi_+, (\chi^{-1})^+\}$ and $\{\Phi_-, (\chi^{-1})^-\}$ the action can be read as the product of these sectors. Furthermore the Q-transformation respects this splitting, i.e. Q does not mix the two sectors:

$$
\begin{aligned}
Q(B_\pm)_{\mu\nu} &= (\psi^1_\pm)_{\mu\nu}\,, \\
Q(\psi^1_\pm)_{\mu\nu} &= -2\partial_{[\mu}(\phi^2_\pm)_{\nu]}\,, \\
Q(\phi^2_\pm)_\mu &= \partial_\mu \phi^3_\pm\,, \\
Q(\phi^{-2}_\pm)_\mu &= (\eta^{-1}_\pm)_\mu\,, \\
Q(\eta_\pm)^{-1}_\mu &= 0\,, \\
Q\phi^{-3}_\pm &= -\partial^\mu(\phi^{-2}_\pm)_\mu\,, \\
QX^1_\pm &= -\partial^\mu(\phi^2_\pm)_\mu\,, \\
Q\chi^{-1,\pm}_{\mu\nu\rho} &= (F^{(\pm)}_\pm)_{\mu\nu\rho}\,.
\end{aligned}
\tag{3.15}
$$

3.2 The s-symmetry

In order to be complete we review also the gauge fixing procedure for the s-transformation of eq. (2.11) implying the existence of the corresponding anti-ghost fields and multiplier fields (see also table 3)

$$
\begin{aligned}
\underline{s}V^{-1}_\mu &= b_\mu\,, & \underline{s}b_\mu &= 0\,, \\
\underline{s}m^{-2} &= \beta^{-1}\,, & \underline{s}\beta^{-1} &= 0\,,
\end{aligned}
\tag{3.16}
$$

		$B_{\mu\nu}$		
	V^{-1}_μ		V^1_μ	
m^{-2}		n		m^2

Table 3: Ghosts and anti-ghosts corresponding to the s-symmetry.

where the vector field V^{-1}_μ exhibits the usual reducible gauge symmetry. This entails the existence of an anti-ghost field n with a multiplier β^1

$$
\underline{s}n = \beta^1\,, \qquad \underline{s}\beta^1 = 0\,. \tag{3.17}
$$

The s-gauge-fixed action becomes therefore

$$
\begin{aligned}
\Sigma_{gf} = \int d^6x\; \underline{s}\Big\{ & V^{-1}_\mu \left(\partial_\nu B^{\mu\nu} - \frac{\gamma_1}{2} b_\mu\right) + m^{-2}\left(\gamma_2\beta^1 + \partial^\mu V^1_\mu\right) + \\
& + n\left(\gamma_3\beta^{-1} + \partial^\mu V^{-1}_\mu\right) - \text{c.c.}\Big\}\,.
\end{aligned}
\tag{3.18}
$$

Choosing $\gamma_1 = -1$ and $\gamma_2 = \gamma_3 = 0$ this is the gauge-fixed action of [1]. On the other hand if we integrate out the multiplier fields b_μ and $\beta^{\pm 1}$ with their algebraic equations of motion

$$
\begin{aligned}
b_\mu &= \frac{1}{\gamma_1}\left(\partial^\nu B_{\mu\nu} - \partial_\mu n\right)\,, \\
\beta^{\pm 1} &= \mp \frac{1}{\gamma_2 - \gamma_3}\partial^\mu V^{\pm 1}_\mu\,,
\end{aligned}
\tag{3.19}
$$

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	ϕ^{-3}	η^{-2}	m^{-2}	β^{-1}	ϕ_μ^{-2}	η_μ^{-1}	$\chi_{\mu\nu\rho}^{-1}$	$H_{\mu\nu\rho}$
dim	4	4	4	4	3	3	3	3
ΦΠ	-3	-2	-2	-1	-2	-1	-1	0
	V_μ^{-1}	b_μ	L_μ	η_μ^1	n	β^1	X^1	η^2
dim	3	3	3	3	2	2	2	2
ΦΠ	-1	0	0	1	0	1	1	2

Table 4: Faddeev-Popov charges for the Lagrange multipliers and anti-ghosts.

and the selection of $\gamma_1 = 1$ and $\gamma_2 - \gamma_3 = -1$ we find

$$\Sigma_{gf} = \int d^6x \left\{ \frac{1}{2} |\partial^\nu B_{\mu\nu}|^2 - m^{-2}\partial^2 m^2 + V_\mu^{-1}\partial^2 V^{1\mu} - \right.$$
$$\left. -\frac{1}{2} n\partial^2 n - V_\mu^{-1}\partial_\nu \psi^{1\mu\nu} + m^{-2}\partial^\mu \phi_\mu^2 - \text{c.c.} \right\} . \tag{3.20}$$

By construction, the complete and fully gauge-fixed action

$$\Sigma_{total} = -\Sigma_{cl} + \Sigma_Q + \Sigma_{gf} \,, \tag{3.21}$$

is invariant under the action of $\underline{s} = s + Q$.

The dimensions and ΦΠ-charges of the anti-ghost and multiplier fields are given in table 4. Finally we want to mention, that the number of bosonic degrees of freedom equals the number of fermionic degrees of freedom. This terminates the review of the topological model presented in [1].

4. The topological linear vector supersymmetry

It is well known that topological field models of Schwarz type are also characterized by the fact that these models posses an additional symmetry: the topological linear vector supersymmetry, which is responsible for the perturbative finiteness of the model. In [4] is shown that this topological linear vector SUSY is also present in the topological Yang-Mills theory in four dimensional space-time if one uses a linear gauge condition for the shift ghost ψ_μ^1 instead of the non-linear gauge condition of this Witten-type model. However, one knows that this model is not perturbative finite. The existence of this topological linear vector SUSY improves considerable the discussion of the algebraic renormalization procedure — leading to an anomaly free theory [4]. Additionally, one has to stress that in Schwarz type models quantized in the axial gauge the constraints coming from the vector SUSY allow only tree-graphs which are always perturbatively finite.

It seems to be quite natural to investigate the possibility of the existence of this vector SUSY also for the model under discussion. Usually, the vector SUSY can be

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constructed from the fact that Σ_{cl} is metric-independent, whereas the metric dependence emerges by the gauge-fixing procedure entailing that the energy-momentum tensor can be written as BRST exact expression [7]

$$T_{\mu\nu} = \underline{s}\Lambda_{\mu\nu} \,. \tag{4.1}$$

From eq. (4.1) follows in a standard manner in calculating the divergence of the energy-momentum tensor — and in considering[2]

$$\int d^6x\ \partial^\mu T_{\mu\nu} = \int d^6x\ \sum_i \delta_\nu\phi_i \frac{\delta\Sigma}{\delta\phi_i} \,, \tag{4.2}$$

the existence of the vector SUSY. Due to the fact that the model in question is a free abelian one, no external unquantized sources are needed in order to describe the non-linear pieces of $\underline{s}$. This implies that the δ_μ-symmetry is not broken linearly by the quantum fields as it is the case for interacting topological field models. Usually, the BRST-operator and the δ_μ-operation closes on-shell on space-time translations

$$\{\underline{s}, \delta_\alpha\} = \partial_\alpha + \text{equations of motion} \,. \tag{4.3}$$

However, for the model under consideration the Wess-Zumino type algebra (4.3) is an off-shell relation. (4.3) implies that the δ_μ-symmetry is in some sense the inverse of the BRST-symmetry.

For the explicit construction of the vector SUSY we have chosen a general ansatz which coefficients are determined by the algebra (4.3) and the Ward identity (4.2). This places also restrictions on the gauge parameters. In particular, we have to set $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\gamma_i = 0$ to obtain a model which is invariant under the vector SUSY. The transformations of the gauge field and the ghosts then are

$$\begin{aligned} \delta_\alpha B_{\mu\nu} &= 0\,, & \delta_\alpha V^1_\mu &= 0\,, \\ \delta_\alpha \psi^1_{\mu\nu} &= \partial_\alpha B_{\mu\nu}\,, & \delta_\alpha \phi^2_\mu &= \partial_\alpha V^1_\mu\,, \\ \delta_\alpha \phi^3 &= \partial_\alpha m^2\,, & \delta_\alpha m^2 &= 0\,, \end{aligned} \tag{4.4}$$

whereas the anti-ghosts and multipliers transform as

$$\begin{aligned} \delta_\alpha X^1 &= \partial_\alpha n\,, & \delta_\alpha \eta^2 &= \partial_\alpha X^1 - \partial_\alpha \beta^1\,, \\ \delta_\alpha V^{-1}_\mu &= \partial_\alpha \phi^{-2}_\mu\,, & \delta_\alpha b_\mu &= \partial_\alpha V^{-1}_\mu - \partial_\alpha \eta^{-1}_\mu\,, \\ \delta_\alpha n &= 0\,, & \delta_\alpha \beta^1 &= \partial_\alpha n\,, \\ \delta_\alpha \phi^{-2}_\mu &= 0\,, & \delta_\alpha \eta^{-1}_\mu &= \partial_\alpha \phi^{-2}_\mu\,, \\ \delta_\alpha \chi^{-1}_{\mu\nu\rho} &= 0\,, & \delta_\alpha H_{\mu\nu\rho} &= \partial_\alpha \chi^{-1}_{\mu\nu\rho}\,, \\ \delta_\alpha m^{-2} &= -\partial_\alpha \phi^{-3}\,, & \delta_\alpha \beta^{-1} &= \partial_\alpha m^{-2} + \partial_\alpha \eta^{-2}\,, \\ \delta_\alpha \phi^{-3} &= 0\,, & \delta_\alpha \eta^{-2} &= \partial_\alpha \phi^{-3}\,. \end{aligned} \tag{4.5}$$

One deduces that for each BRST-doublet there exists also a corresponding δ_α-doublet. It is straightforward to prove that $\delta_\alpha\left(\Sigma_{cl} + \Sigma_Q + \Sigma_{gf}\right) = 0$ is indeed valid.

[2] ϕ_i stands collectively for all fields characterizing the topological field model.

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5. Twist

In a next step we try to reconstruct the topological shift and vector-supersymmetry transformations of the fields through a twist of a supersymmetry multiplet. Hence, we need a tensor multiplet containing a second-rank antisymmetric tensor gauge field. In six dimensions this is given by a (2,0) tensor multiplet which consists of the desired tensor $\tilde{B}_{\mu\nu}$, a real field Φ_{ij} and a spinor $\kappa_{\alpha i}$.

The spinors in six dimensions transform under $\mathrm{Spin}(1,5)\times\mathrm{Spin}(5)$, where Spin(5) is isomorphic to USp(4). USp(4) also provides the existence of symplectic Majorana-Weyl spinors. The symplectic Majorana-Weyl condition is

$$\kappa^*_{\dot{\alpha}i} = B_{\dot{\alpha}}{}^{\beta}\Omega^{ij}\kappa_{\beta j} \,, \tag{5.1}$$

where $\alpha = 1,2,3,4$ are the spinor indices which transform under Weyl-projected Spin(1,5), whereas $i = 1,2,3,4$ denote internal indices transforming under USp(4). The metric Ω^{ij} is real and antisymmetric and allows raising and lowering of the internal indices by $\kappa^i = \Omega^{ij}\kappa_j$ respectively $\kappa_i = \kappa^j\Omega_{ji}$. The matrix B is unitary and acts like $\psi_{\dot{\alpha}} = B^{\beta}_{\dot{\alpha}}\psi_\beta$ and $\psi^\alpha = \psi^{\dot{\beta}}B^{\alpha}_{\dot{\beta}}$.

With the constraint $\Omega^{ij}\Phi_{ij} = 0$ this yields three respectively five bosonic on-shell degrees of freedom for $\tilde{B}_{\mu\nu}$ and Φ_{ij}, and eight fermionic on-shell degrees of freedom for $\kappa_{\alpha i}$ [8].

In order to perform the topological twist [2] we have to break down SO(1,5) to SO(5), since SO(5) is locally isomorphic to USp(4). Now the twist is done by choosing the diagonal subgroup of the symmetry group $\mathrm{SO}(5)\times\mathrm{USp}(4)$. The invariant axis under SO(5) will be denoted by $n^\mu = (1,\vec{0})$ in the vector representation. For the spin-representation the vector n^μ is translated to $n_\mu\,(\bar{\gamma}^\mu)^{\alpha\beta}$. The Spin(5) sub-group consists of the rotations of Spin(1,5) that do not change $n_\mu\,(\bar{\gamma}^\mu)^{\alpha\beta}$. Now technically diagonalizing $\mathrm{SO}(5)\times\mathrm{USp}(4)$ is done by identifying the respective indices $(\alpha\to i,\ldots)$. Some care is in order, as a metric, similar to Ω_{ij} of USp(4) is missing so far. As a metric of Spin(5) $n_\mu\,(\bar{\gamma}^\mu)^{\alpha\beta}$ has to be chosen, as it is invariant by definition and any different metric would further constrain the set of rotations. Identifying indices forces one to identify the matrices as well, so one has to set

$$n_\mu\,(\bar{\gamma}^\mu)^{\alpha\beta} \to n_\mu\,(\bar{\gamma}^\mu)^{ij} = \Omega^{ij} \,. \tag{5.2}$$

The supersymmetry action on the components of the complexified multiplet is given by[3]

$$\begin{aligned}
\delta\Phi_{jk} &= \varepsilon^{\alpha}_{[j}\kappa_{\alpha k]} + \frac{1}{4}\Omega_{jk}\Omega^{il}\varepsilon^{\alpha}_{l}\kappa_{\alpha i} \,,\\
\delta\bar{\Phi}^{jk} &= -\bar{\varepsilon}^{\dot{\alpha}[j}\bar{\kappa}^{k]}_{\dot{\alpha}} - \frac{1}{4}\Omega^{jk}\Omega_{il}\bar{\varepsilon}^{\dot{\alpha}l}\bar{\kappa}^{i}_{\dot{\alpha}} \,,
\end{aligned}$$

[3]The analytic continuation is understood formally since it breaks the Schouten identities which provide the closure of the algebra [8].

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$$
\begin{aligned}
\delta\kappa_{\alpha i} &= \tilde{F}^{+}_{\mu\nu\rho}\,(\gamma^{\mu\nu\rho})_{\alpha\dot{\beta}}\,\varepsilon^{\dot{\beta}}_{i} + (\gamma^{\mu}\partial_{\mu})_{\alpha\dot{\beta}}\,\Phi_{ij}\bar{\varepsilon}^{\dot{\beta}j}\,,\\
\delta\bar{\kappa}^{i}_{\dot{\alpha}} &= \bar{\varepsilon}^{\beta i}\bar{\tilde{F}}^{+}_{\mu\nu\rho}\,(\gamma^{\mu\nu\rho})_{\beta\dot{\alpha}} + \varepsilon^{\beta}_{j}\,(\gamma^{\mu}\partial_{\mu})_{\beta\dot{\alpha}}\,\bar{\Phi}^{ij}\,,\\
\delta\tilde{B}_{\mu\nu} &= \bar{\kappa}^{i}_{\dot{\alpha}}\,(\gamma_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}\,\varepsilon^{\dot{\beta}}_{i}\,,\\
\delta(\bar{\tilde{B}})_{\mu\nu} &= \bar{\varepsilon}^{\alpha i}\,(\gamma_{\mu\nu})^{\ \beta}_{\alpha}\,\kappa_{\beta i}\,.
\end{aligned}
\tag{5.3}
$$

By an expansion with respect to the gamma matrices the spinors $\kappa_{\alpha i}$ and $\bar{\kappa}^{i}_{\dot{\alpha}}$ can be written as

$$
\begin{aligned}
\kappa_{\alpha i} &= (\gamma^{\mu\nu\rho})_{\alpha i}\,(\chi^{-1,+})_{\mu\nu\rho} + (\gamma^{\mu})_{\alpha i}\,(\eta^{-1}_{+})_{\mu}\,,\\
\bar{\kappa}^{\ i}_{\dot{\alpha}} &= (\gamma^{\mu\nu})^{\ i}_{\dot{\alpha}}\,(\psi^{1}_{+})_{\mu\nu} + B^{\ i}_{\dot{\alpha}}X^{1}_{+}\,,
\end{aligned}
\tag{5.4}
$$

whereas the scalars are

$$
\Phi_{ij} = (\gamma^{\mu})_{ij}\,(\phi^{-2}_{+})_{\mu}\,, \qquad \bar{\Phi}^{ij} = (\bar{\gamma}^{\mu})^{ij}\,(\phi^{2}_{+})_{\mu}\,.
\tag{5.5}
$$

The fields $\tilde{B}_{\mu\nu}$ and $\bar{\tilde{B}}_{\mu\nu}$ recombine to the gauge field $(B_{+})_{\mu\nu}$

$$
(B_{+})_{\mu\nu} = -\frac{1}{8}\left(\tilde{B}_{\mu\nu} + \bar{\tilde{B}}_{\mu\nu}\right).
\tag{5.6}
$$

The twist yields for the topological shift symmetry $\tilde{Q}$

$$
\begin{aligned}
\tilde{Q}(\phi^{-2}_{+})_{\mu} &= -P_{\mu\nu}\eta^{-1\nu}_{+}\,,\\
\tilde{Q}(\phi^{2}_{+})_{\mu} &= 0\,,\\
\tilde{Q}(\chi^{-1,+})_{\mu\nu\rho} &= -(F_{+})^{+}_{\mu\nu\rho}\,,\\
\tilde{Q}(\eta^{-1}_{+})_{\mu} &= 0\,,\\
\tilde{Q}X^{1}_{+} &= -\partial_{\mu}(\phi^{2}_{+})^{\ \mu}\,,\\
\tilde{Q}(\psi^{1}_{+})_{\mu\nu} &= -\partial_{[\mu}(\phi^{2}_{+})_{\nu]}\,,\\
\tilde{Q}(B_{+})_{\mu\nu} &= (\psi^{1}_{+})_{\mu\nu}\,,
\end{aligned}
\tag{5.7}
$$

where $P_{\mu\nu}$ projects to the five dimensional subspace $P_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}$. It has been emphasized in [1] that one has to add a "topological package" of scalar fields of opposite statistics $(\phi^{\pm 3}, (\phi^{\pm 2})_0)$ to match the six dimensionally covariant form of (3.15). We also emphasize that in order to get the field content of the Q-fixed action in section (3.1) we have to take also a twisted $(0,2)$ multiplet, which gives us the anti-self dual part.

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We also can find a vector transformation from the twist, which we denote by $\tilde{\delta}_\alpha$ [4].

$$
\begin{aligned}
\tilde{\delta}_\alpha(\phi_+^{-2})_\mu &= 0\,, \\
\tilde{\delta}_\alpha(\phi_+^{2})_\mu &= -2g^{\nu\rho}P_{\mu\nu}(\psi_+^1)_{\alpha\rho} - P_{\alpha\mu}X_+^1\,, \\
\tilde{\delta}_\alpha(\psi_+^1)_{\mu\nu} &= -3!(\bar{F}_+)^+_{\alpha\mu\nu}\,, \\
\tilde{\delta}_\alpha X_+^1 &= 0\,, \\
\tilde{\delta}_\alpha(\eta_+^{-1})_\mu &= -\left(g_{\alpha\mu}\partial^\rho(\phi_+^{-2})_\rho - \partial_\mu(\phi_+^{-2})_\alpha - \partial_\alpha(\phi_+)_\mu^{-2}\right)\,, \\
\tilde{\delta}_\alpha\chi_{\mu\nu\rho}^{-1,+} &= -\frac{1}{2}\left(g_{\alpha[\mu}\partial_\nu(\phi_+^{-2})_{\rho]} - \frac{1}{3!}\varepsilon_{\alpha\mu\nu\rho\kappa\lambda}\partial^\kappa(\phi_+^{-2})^\lambda\right)\,, \\
\tilde{\delta}_\alpha(B_+)_{\mu\nu} &= 3!(\chi^{-1,+})_{\alpha\mu\nu} - g_{\alpha[\mu}(\eta_+^{-1})_{\nu]}\,.
\end{aligned}
\tag{5.8}
$$

In [5] similar results for the shift symmetry and the vector supersymmetry were obtained in the case of Witten's topological Yang-Mills theory by twisting the $N=2$ supersymmetric algebra. However, an analysis of the results of [5] shows that they differ from the results given in [4]. This is due to the fact that the twist procedure starts with a supersymmetric multiplet in the Wess-Zumino-gauge. The super-Poincaré algebra closes on translations only modulo gauge-transformations and equations of motion. Hence, the twist leads to an algebra that looks like

$$\{\tilde{Q},\tilde{\delta}_\alpha\} = \partial_\alpha + \text{equations of motion} + \text{gauge-transformations}\,, \tag{5.9}$$

where we denoted the twisted vector supersymmetry by $\tilde{\delta}_\alpha$. The model which one obtains, is not totally gauge-fixed, since the gauge-degree of freedom for A_μ is left [2]. We expect that the twist in a manifestly $N=2$ supersymmetric formulation would give a result as in [4]. In this way the existence of a vector supersymmetry in the gauge fixed $N=2$ SYM can be motivated.

Let us contrast this now to our findings in the six dimensional model at hand. The vector operator (5.8) can be constructed in rather analogous way as in the four dimensional SYM. Its properties are however rather different. We find that $\{\tilde{Q},\tilde{\delta}_\alpha\}$ does not close on translations modulo gauge transformations and equations of motion. This is because the six dimensional supersymmetry makes use of Schoutens identities (see e.g. [8]) valid for symplectic Majorana-Weyl spinors. The latter property is however broken by the twist, as already noted in [1]. In order to connect the twisted $(2,0)$ multiplet with the field content of the topological model one has also to introduce the "topological package" $(\phi^{\pm3},(\phi^{\pm2})_0)$. One might hope that these fields can compensate the terms breaking the Schoutens identities. However one gets disappointed. It is not possible to incorporate $\phi^{\pm3}$ such that the algebra closes even on these fields. Lorentz symmetry and ghost number conservation make it impossible.

[4] $\tilde{Q}$ and $\tilde{\delta}_\alpha$ are obtained by using analogous decompositions as in (5.4) for the supersymmetry parameters. This implies that there are also some tensorial transformations. We do not further investigate these.

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Therefore, contrary to the case in four dimensions, the existence of a topological vector SUSY we found in the previous chapter seems rather unrelated with the fact that part of the BRST-operator can be constructed by a twist of the six dimensional $(2,0)$ multiplet.

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